FINE STRUCTURE OF THE ZEROS OF ORTHOGONAL POLYNOMIALS: A REVIEW

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We review recent work on zeros of orthogonal polynomials.

1. Introduction

Zeros of orthogonal polynomials have had a fascination at least since Gauss' discovery that optimal quadrature for the Riemann integral on $[-1,1]$ involves the zeros of the Legendre polynomials. A special reason for recent interest concerns the fact that zeros are eigenvalues of cutoff finite difference matrices.

Explicitly, if P_n, p_n are the monic orthogonal and orthonormal polynomials for OPRL (RL = real line) and Φ_n, φ_n for OPUC (UC = unit circle), then

$$P_n(x) = \det(x - \pi_n M_x \pi_n) \tag{1.1}$$

$$\Phi_n(z) = \det(z - \pi_n M_z \pi_n) \tag{1.2}$$

where π_n is the projection onto the n-dimensional space of polynomials of degree at most $(n-1)$ and M_x (resp. M_z) is multiplication by x (resp. z) on $L^2(\mathbb{R}, d\rho)$ (resp. $L^2(\partial\mathbb{D}, d\mu)$).

By using the $\{p_j\}_{j=0}^{n-1}$ basis, $\pi_n M_x \pi_n$ can be replaced by a cutoff Jacobi matrix, and by using a cutoff CMV matrix, $\pi_n M_z \pi_n$ can be replaced by a cutoff CMV basis (see Chapter 4 of [1]). In general, we follow the conventions of [1, 2] throughout this article. In particular,

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$\{a_n, b_n\}_{n=1}^{\infty}$ are the Jacobi parameters, $\{\alpha_n\}_{n=0}^{\infty}$ the Verblunsky coefficients, and $Q_n^*(z) = z^n \overline{Q_n(1/\bar{z})}$ the Szegő dual.

We can also describe paraorthogonal polynomials (POPUC) (see [3]) in terms of (1.1)/(1.2). $\pi_n M_z \pi_n$ is norm-preserving on $\operatorname{Ran} \pi_{n-1}$, the polynomials of degree at most $n-2$, a space of codimension 1 in $\operatorname{Ran} \pi_n$. There is thus a one-parameter family, $\{C(\beta) \mid \beta \in \partial\mathbb{D}\}$, of unitary modifications of $\pi_n M_z \pi_n$ obtained by taking $\operatorname{Ran} \pi_n \cap [\operatorname{Ran} \pi_{n-1}]^\perp$ (i.e., multiples of φ_{n-1}) to vectors in $\operatorname{Ran} \pi_n \cap \operatorname{Ran}(z\pi_{n-1})$. One can show

$$\det(z - C(\beta)) = \Phi_{n-1}(z) - \beta \Phi_{n-1}^*(z) \tag{1.3}$$

which defines the POPUC.

In the past two years, I (partly jointly with Brian Davies and Yoram Last) have been looking at the fine structure of the zeros of orthogonal polynomials [4, 5, 6, 7, 8] as has my student, Mihai Stoiciu, in his thesis [9, 10]. It is this work that I want to review here.

Earlier work (see Section 2) established that, in many cases, the bulk of the zeros of OP's approach a canonical density, often uniform density on a circle in the OPUC case and the measure for (first kind) Chebyshev polynomials in the OPRL case. When, motivated by earlier pictures of Saff [11], I prepared pictures of zeros for my book [1, pp. 414–423], I was struck by two kinds of regularity shown in Figures 1 and 2.

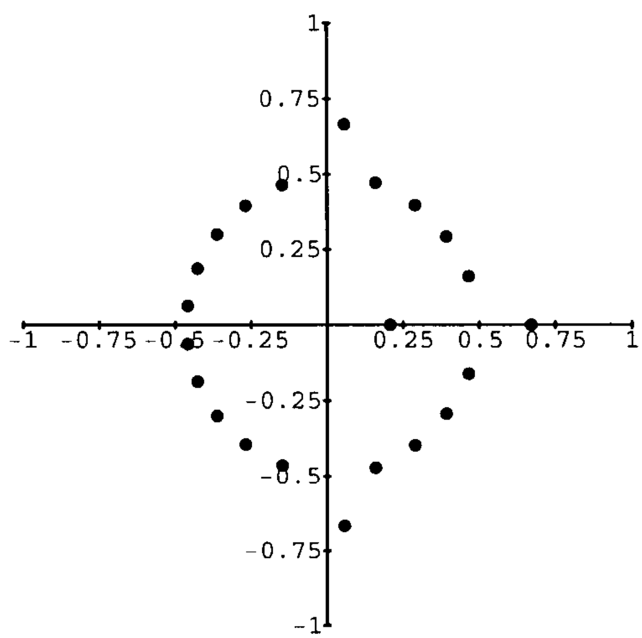

Figure 1.

Figure 1 (taken from [5]) shows the zeros of Φ_{22} when

$$\alpha_n = \left(\frac{1}{2}\right)^{n+1}\left(1 + 2\cos\left(\frac{\pi}{2}(n+1)\right)\right) \tag{1.4}$$

a somewhat complicated example to show features I will discuss later. For now, I will focus on the eighteen zeros very near $|z| = \frac{1}{2}$. At first sight, they do seem to be converging to a uniform distribution. In fact, the approach to uniformity is strikingly strong — they seem to be equally spaced.

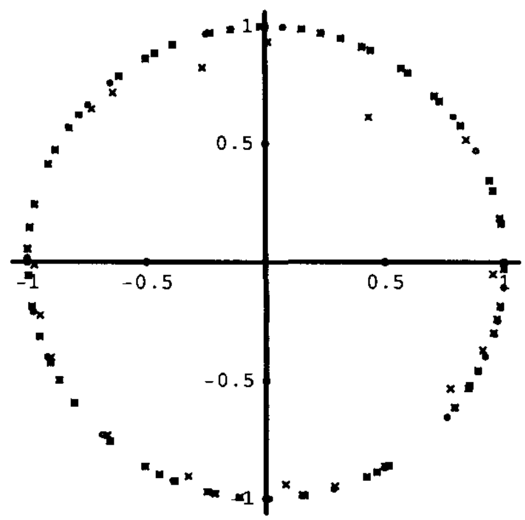

Figure 2.

Figure 2, kindly prepared for me by Mihai Stoiciu, shows zeros for OPUC/POPUC of degree 70 with $\alpha_0, \dots, \alpha_{68}$ chosen randomly and independently, according to a uniform distribution in $\{z \mid |z| < \frac{1}{2}\}$. The diagonal crosses show the zeros of OPUC with α_{69} also having this distribution. The circles show the zeros for the POPUC where β is chosen uniformly in $\partial\mathbb{D}$. In many cases, they appear as a cross upon a circle. Of course, a single choice is made using Mathematica's random number generator, so this is a "typical" choice from a random ensemble. Theorems assert that again the bulk of zeros converge for either OPUC or POPUC to a uniform density on $\partial\mathbb{D}$. At first sight, this seems questionable — look at the clumping and gaps! But, in fact, this distribution is "regular": 70 points placed at random around the circle would show similar clumps and gaps!

Thus, the main theme of the work I will describe is clock (strict equal spacing in the limit) behavior for one set of parameters and Poisson distribution for another.

Neither idea is totally new. Sixty-five years ago, Erdös–Turán [12] discussed $O(1/n)$ upper and lower bounds and even proved clock behavior under some strong global hypotheses. In the 1990's, Vértesi [13, 14, 15] proved clock spacing for the zeros of Jacobi polynomials.

As far as Poisson distributions of zeros, Molchanov [16] first proved an analog for certain random Schrödinger operators, and Minami [17] for some discrete models that overlap random OPRL. There are, however, new technical issues that need to be addressed to get the random results for POPUC [9, 10] and OPUC [8].

Before getting into the details, it is worth making a few remarks:

1. A well-known idea in quantum physics is that the levels repel each other as parameters are varied (see, e.g., [18, 19, 20]). Equal spacing is an extreme form of eigenvalues trying to stay as far away from each other as possible.

2. The Poisson result can only hold in the random case because levels do not repel — this is connected to localization in the random model (see Section 12.6 of [2]).

3. There are still open questions especially for OPUC; see the discussion of open questions in [7].

4. Fine structure of the zeros is intimately connected to suitable asymptotics for the OP's.

5. There is a third distribution of eigenvalue spacing that occurs, but not, so far, for OP's, namely those for random matrices [21]. It is intermediate to clock and Poisson. Indeed, GUE and GOE are distinct and both lie between. In fact, there is a one-parameter family (β-distributions) that all lie between. We discuss this further in Section 7.

In Section 2, we provide some background on previous work that is related to the subjects we study here. We are not trying to be comprehensive in reviewing all aspects of previous work on zeros which is discussed in Saff [11] and in Sections 1.7 and Chapter 8 of [1]. In Section 3, we discuss OPUC when the α_n are a sum of competing exponentials and an error exponentially smaller. In Sections 4 and 5, we discuss some OPRL and POPUC in the Nevai class and perturbed periodic case, respectively. In Section 6, we discuss $O(1/n)$ bounds. In Section 7, we discuss Stoiciu's work on random POPUC, and in Section 8, the linear variational principle of Davies–Simon and the extension of Stoiciu's results to OPUC.

2. Prior Work

Here we mention earlier results that set the stage for later sections. The approximate density of zeros, $d\nu_n$, is the pure point probability measure giving weight k/n to a zero of multiplicity k (for OPRL and POPUC, $k = 1$). If $d\nu_n$ has a limit, we denote it $d\nu$ and call it the density of zeros. It is certainly not automatic that a limit exists. Simon–Totik [22] have given examples of OPUC where the set of limit points of $d\nu_n$ is all measures on $\overline{\mathbb{D}}$! But under mild regularity conditions, $d\nu_n$ does converge:

Theorem 2.1. *In the OPRL case, if $b_n \to 0$, $a_n \to 1$, then $d\nu_n$ converges to*

$$d\nu(x) = \pi^{-1}(4 - x^2)^{-1/2}\chi_{[-2,2]}(x)\,dx \tag{2.1}$$

Theorem 2.2. *Let $\{a_n(\omega), b_n(\omega)\}_{n=1}^{\infty}$ be given by an ergodic process. Then for a.e. ω, $d\nu_n^{\omega}$ converges to a limit $d\nu$ that is ω-independent.*

Remarks. 1. If $b_n \equiv 0$, $a_n \equiv 1$, then $P_n = c_n \sin((n+1)\theta)/\sin(\theta)$ where $x = 2\cos\theta$ and $d\nu(x) = d\theta/\pi$, "explaining" the form of (2.1)

2. Results of the form Theorem 2.1 go back to Erdös–Turán [12]. It was Nevai [23] who realized all that was needed was $b_n \to 0$, $a_n \to 1$ (called the Nevai class). This OPRL work looks at $a_n \to \frac{1}{2}$ rather than $a_n \to 1$.

3. Both theorems appear in the physics/Schrödinger operator literature; see [24, 25, 26].

4. The ideas behind the proofs are quite simple: By (1.1), $\int x^{\ell}\,d\nu_n = \frac{1}{n}\mathrm{Tr}(J_{n;F}^{\ell})$ where $J_{n;F} = \pi_n M_x \pi_n$ realized as a cutoff Jacobi matrix. By uniform boundedness of the support of $\{d\nu_n\}$, it suffices to prove convergence of the moments, and that is immediate in the case of Theorem 2.1 and follows from the Birkhoff ergodic theorem in the case of Theorem 2.2.

For OPUC and POPUC, the limit results are

Theorem 2.3. *If $\lim|\alpha_n|^{1/n} = R^{-1} \le 1$ and if $\frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j| \to 0$, then $d\nu_n$ for OPUC converges to $d\theta/2\pi$ concentrated on the circle of radius R^{-1}.*

Theorem 2.4. *If $\{\alpha_n(\omega)\}_{n=0}^{\infty}$ is given by an ergodic process and $\mathbb{E}(\log|\alpha_n(\omega)|^{-1}) < \infty$, then for a.e. ω, $d\nu_n^{\omega}$ for OPUC has an ω-independent limit supported on $\partial\mathbb{D}$.*

Theorem 2.5. *If $\frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j| \to 0$, $d\nu_n$ for POPUC converges to $d\theta/2\pi$ on $\partial\mathbb{D}$.*

Theorem 2.6. *If $\{\alpha_n(\omega)\}_{n=0}^{\infty}$ is given by an ergodic process, $d\nu_n^\omega$ for POPUC converges for a.e. ω to an ω-independent limit $d\nu$. If also $\mathbb{E}(\log|\alpha_n(\omega)|^{-1}) < \infty$, then this limit is the same for OPUC and POPUC.*

Remarks. 1. Theorem 2.3 is due to Mhaskar–Saff [27], with a partially alternate proof using CMV matrices in Simon [1]. Theorem 2.4 is Theorem 10.5.19 of [2]. Theorem 2.5 is implicit in Golinskii [28]. Theorem 2.6 is an easy consequence of the ergodic theorem (see Remark 3 below).

2. Golinskii [28] also discusses the case where $\frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j - a| \to 0$ for some $a \in \mathbb{D}$. The density of zeros in this case is the equilibrium measure for an arc. Indeed, density of zeros are often equilibrium measures (see Stahl–Totik [29] and Chapter 11 of [2]).

3. As in the OPRL case, positive moments of $d\nu_n$ are given by traces of powers of cutoff CMV matrices. For POPUC, where measures live on $\partial\mathbb{D}$ and positive moments determine the measure, this is all that is needed for convergence of $d\nu_n$. But for OPUC, one needs a separate argument that shows $d\nu_n$ is asymptotically supported on a circle. In the ergodic case, this is easy. In the case of Theorem 2.3, one uses that the product of the zeros is $\pm\bar{\alpha}_n$ and Theorem 2.7 below.

4. The cutoff CMV matrices for POPUC and OPUC differ in only two rows, so moments of the $d\nu_n$ for these two cases have the same limits. If OPUC has a limit supported on $\partial\mathbb{D}$, POPUC has the same limit.

We mention two earlier results for OPUC that go beyond the bulk results of Theorems 2.1–2.6:

Theorem 2.7. *If $\limsup|\alpha_n|^{1/n} = R^{-1} < 1$, then the inverse of the Szegő function $D(z)^{-1}$ has an analytic continuation to $\{z \mid |z| < R\}$. A point z_0 in $\{z \mid R^{-1} < |z| < 1\}$ is a limit point of zeros of $\Phi_n(z)$ if and only if $D(\bar{z}_0^{-1})^{-1} = 0$.*

Theorem 2.8. *If $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = b$ and $\lim_{n\to\infty}|\alpha_n| = 0$, then for large n, $\Phi_n(z)$ has no zeros in $\{z \mid |z| < b - \varepsilon\}$ for each $\varepsilon > 0$.*

Remarks. 1. Theorem 2.7 is due to Nevai–Totik [30] and the zeros asymptotically outside $|z| = R^{-1}$ are the Nevai–Totik zeros. Figure 1 shows three Nevai–Totik zeros.

2. Theorem 2.8 is due to Barrios–López–Saff [31] who also treat cases where, for $|b| < 1$, $\alpha_n b^{-n}$ approaches a periodic sequence.

3. OPUC With Competing Exponential Decay

Here, following [4, 5], we want to discuss zeros of OPUC where Verblunsky coefficients have the form

$$\alpha_n = \sum_{\ell=1}^{L} C_\ell b_\ell^n + O((b\Delta)^n) \tag{3.1}$$

where $|b_\ell| = b < 1$ for $\ell = 1, \dots, L$ and $\Delta < 1$. It is known ([31] for $L = 1$, [1] for general L; see also [32, 33, 34]) that (3.1) is equivalent to:

$$\begin{gathered} D(z)^{-1} \text{ is meromorphic in } \{z \mid |z| < b^{-1} + \varepsilon\} \\ \text{with poles exactly at } \{\bar{b}_\ell^{-1}\}_{\ell=1}^{L} \end{gathered} \tag{3.2}$$

We want to describe a complete asymptotic analysis of the zeros of $\Phi_n(z)$ for n large. Related results were found independently by Martínez-Finkelshtein, McLaughlin, and Saff [33]. We already know the zeros outside $|z| = b$ are the Nevai–Totik zeros which, by (3.2), are finite in number. Here, from [5], is what happens near the circle of radius b:

Theorem 3.1. *If (3.1) holds, then for some $\delta > 0$, all zeros of $\Phi_n(z)$ in $\{z \mid b - \delta < |z| < b + \delta\}$ lie in an annulus of width $O(\log n/n)$ about $|z| = b$, and for n large, they can be ordered via an increasing argument $\{z_j^{(n)}\}_{j=1}^{N_n}$ where $|N_n - n|$ is bounded. We have*

$$\left|\frac{z_{j+1}}{z_j}\right| = 1 + O\left(\frac{1}{n \log n}\right) \tag{3.3}$$

Moreover, if $\{\tilde{z}_j^{(n)}\}_{j=1}^{N_n+L}$ are $z_j^{(n)}$ with the $\{\bar{b}_\ell\}_{\ell=1}^{L}$ inserted, and if $\tilde{z}_{N_n+L+1}^{(n)} = \tilde{z}_1^{(n)} + 2\pi$, then

$$\arg \tilde{z}_{k+1}^{(n)} - \arg \tilde{z}_k^{(n)} = \frac{2\pi}{n} + O\left(\frac{1}{n \log n}\right) \tag{3.4}$$

for $k = 1, 2, \dots, N_n + L$.

Remarks. 1. The errors in Theorem 3.1 are global and can be strengthened away from those finite number of points on $\{z \mid |z| = b\}$ where $D(1/\bar{z})^{-1} = 0$. If there are no zeros of $D(w)^{-1}$ on $\{w \mid |w| = b^{-1}\}$, then the improvements are global. The improvements replace $O(1/n \log n)$ by $O(1/n^2)$ and $O(\log n/n)$ by $O(1/n)$.

2. Basically, the zeros are equally spaced except for gaps at $z = \bar{b}_\ell$, $\ell = 1, 2, \dots, L$. This is clearly seen in Figure 1.

3. The proofs depend on detailed asymptotics near $|z| = b$ which are obtained in [5] by carefully iterating the Szegő recursion on the circle of radius $|z| = b$ and $|z| = b^{-1}$. For $L = 1$, [4] has a different argument analyzing second-order difference equations.

As for zeros inside $|z| = b$, these are controlled by the degree $L - 1$ polynomials (where $\omega_\ell = \bar{b}_\ell/b$),

$$P_n(z) = \sum_{\ell=1}^{L} \bar{C}_\ell \omega_\ell^n \prod_{k \neq \ell} (z - b_k)$$

which are almost periodic in n and when the ω_ℓ's are roots of unity, periodic in n.

Theorem 3.2. *If* (3.1) *holds, for any $\varepsilon > 0$, there is any N so for $n > N$, the zeros of $\varphi_n(z)$ in $\{z \mid |z| < b - \varepsilon\}$ are precisely within ε of the zeros of $P_n(z)$ in this region, where $\varphi_n(z)$ has k zeros within ε of a z_0 with $P_n(z_0) = 0$ where k is the multiplicity of the zero of P_n.*

Figure 1 has $\omega_1, \omega_2, \omega_3 = 1, i, -i$ so P_n is periodic mod 4 and the one zero shown for Φ_{22} with $|z| \ll \frac{1}{2}$ will recur for $\Phi_{26}, \Phi_{30}, \Phi_{34}, \dots$. The zero of P_{22} is at $\frac{1}{2}(\sqrt{2} - 1)$ and the zero shown agrees numerically to 10^{-9}!

Except for the results in Section 8, we will not discuss OPUC further. We note [6] has results on zeros of periodic OPUC. Among the open questions for OPUC, we mention:

Conjecture 3.1. *If $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $|\alpha_n|^{1/n} \to 1$, then the zeros are mainly near $\partial\mathbb{D}$ and have clock behavior away from $z = 1$.*

Conjecture 3.2. *If $\alpha_n \sim Cn^{-\gamma}$ for $\gamma < 1$, then the zeros have a gap of $O(n^{-\gamma})$ near $z = 1$.*

4. Clock Behavior Within the Nevai Class

Anticipated results in the decaying random case (see Section 7) show that the Nevai condition $b_n \to 0$, $a_n \to 1$ is not expected to be sufficient for clock behavior, but a large subclass does have this behavior. Since the density of zeros is given by (2.1), clock behavior in this case cannot mean global equal spacing in the x scale but in a scale given by $d\nu/dx$ (i.e., the $d\theta$ of Remark 1 after Theorem 2.2). We have:

Theorem 4.1. [7] *Suppose that the Jacobi parameters of an OPRL obey*

$$|a_n - 1| + |b_n| \to 0 \qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty \tag{4.1}$$

Then one has clock behavior uniformly on each interval $[-2+\varepsilon, 2-\varepsilon]$ *in the sense that*

$$\lim_{n\to\infty} \left[\left\{ \sup \left| n|x' - x| - \left(\frac{d\nu}{dx}\right)^{-1} \right| \;\middle|\; x, x' \text{ are successive zeros of } p_n \text{ in } [-2+\varepsilon, 2-\varepsilon] \right\} \right] = 0 \tag{4.2}$$

Remarks. 1. (4.1) holds if

$$\sum_{n=1}^{\infty} |b_n| + |a_n - 1| < \infty$$

and also if $b_n = n^{-\alpha}$, $a_n = 1 + n^{-\beta}$ for any $\alpha, \beta > 0$.

2. This result includes results of Vértesi [13, 14, 15] as well as results from [4].

The proof of this result has two elements. One is what Schrödinger operator experts would call Jost asymptotics, although in the context of Jacobi polynomials, they go back to Laplace, Heine, Darboux, and Stieltjes (see Szegő [35]). This oscillatory asymptotics guarantees that there are zeros with correct asymptotic spacing.

The second issue is to ensure that there aren't additional zeros such as $\sin x - \frac{1}{n}\sin(n^2 x)$ has. In [4], this second issue is dealt with awkwardly. The point of [7] is that one can deal with it easily by proving *a priori* $O(1/n)$ lower bounds as discussed in Section 6.

We note that the control of oscillatory solutions only under (4.1) is subtle, using ideas of Kooman [36]; see also Golinskii–Nevai [37] and Section 12.1 of [2].

For POPUC, we have:

Theorem 4.2. [7] *If the Verblunsky coefficients* α_n *of a POPUC obey*

$$|\alpha_n| \to 0 \qquad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \tag{4.3}$$

then for any $\varepsilon > 0$, the POPUC zeros are $2\pi/n$ spaced in that

$$\lim_{n\to\infty} \sup\left[\left\{ n\left[|\theta' - \theta| - \frac{2\pi}{n}\right] \,\middle|\, e^{i\theta}, e^{i\theta'} \text{ successive zeros} \right.\right.$$
$$\left.\left. \text{of a POPUC of degree } n;\ \varepsilon < \theta < 2\pi - \varepsilon \right\}\right] = 0 \tag{4.4}$$

5. Clock Behavior for Periodic OPUC

It is well known (see, e.g., [38, 39, 40, 41]) that if a set of Jacobi parameters, $\{a_n^{(0)}, b_n^{(0)}\}_{n=1}^\infty$, is periodic, that is,

$$a_{n+p}^{(0)} = a_n^{(0)} \qquad b_{n+p}^{(0)} = b_n^{(0)} \tag{5.1}$$

for some p and all n, then the essential spectrum of the corresponding Jacobi matrix, $J^{(0}$, consists of p bands $\cup_{j=1}^p [\alpha_j, \beta_j]$ where $\beta_j \leq \alpha_{j+1}$. Generically, $\beta_j < \alpha_{j+1}$ and there are $p-1$ gaps, but there can be some closed gaps. There is a polynomial Δ of degree p so $\Delta^{-1}[-2,2] = \cup_{j=1}^p [\alpha_j, \beta_j]$ and the density of zeros is given by

$$\Delta(x) = 2\cos(p\pi(1 - k(x))) \tag{5.2}$$
$$dk = \text{density of zeros} \tag{5.3}$$

Already in 1922, Faber [42] knew that dk is the potential theory equilibrium measure for $\cup_{j=1}^p [\alpha_j, \beta_j]$. There is some overlap of the results below and work of Peherstorfer [43, 44, 45]. As far as the strictly periodic case, [6] contains the following elementary but striking result:

Theorem 5.1. *The zeros of P_{mp-1} consist of the zeros of P_{p-1}, which has exactly one point in each gap, and those points $x_{\kappa,q}^{(m)}$, $\kappa = 1, \dots, p$; $q = 1, \dots, m-1$, where*

$$k(x_{\kappa,q}^{(m)}) = \frac{\kappa - 1}{p} + \frac{q}{mp} \tag{5.4}$$

Remarks. 1. Thus, we can precisely give the zeros of P_{mp-1}. The points in the gaps are related to "Dirichlet data" (see [6]). Those in (5.4) are $(m-1)$ points in each band.

2. [6] has two proofs of this result. One can also be based on results of Peherstorfer [43].

3. [6] has additional results on the strictly periodic case and on the OPUC situation which is more subtle.

As for perturbations of the periodic case, [7] has

Theorem 5.2. *Let $a_n^{(0)}, b_n^{(0)}$ obey* (5.1) *and suppose that*

$$\lim_{n\to\infty} |a_n - a_n^{(0)}| + |b_n - b_n^{(0)}| = 0 \tag{5.5}$$

$$\sum_{n=1}^{\infty} |a_{n+p} - a_n| + |b_{n+p} - b_n| < \infty \tag{5.6}$$

Then for any compact subset, S, of $\cup_{j=1}^{p}(\alpha,\beta_j)$ (compact in the interior of the bands), we have

$$\lim_{n\to\infty} \left[\left\{ \sup \left| n|x' - x| - \left(\frac{dk}{dx}\right)^{-1} \right| \;\middle|\; x', x \text{ are successive zeros of } p_n \text{ in } S \right\} \right] = 0 \tag{5.7}$$

6. $1/n$ Bounds

As noted in Section 4, one part of proving of Theorem 4.1 (and also Theorem 5.2) is proving *a priori* $O(1/n)$ lower bounds on eigenvalue spacing. Such upper and lower bounds are weaker than clock behavior but are known to hold in much greater generality and are of much greater antiquity, with techniques due to Erdös–Turán [12], Szegő [35], Nevai [23], and Golinskii [28]. [7] has a compendium of new techniques and refinements of old techniques on this problem. We will discuss one result here and refer the reader to [7] for others. The following is interesting because it only requires information at a single value of x. Let $T_n(x)$ be the transfer matrix for solutions of the difference equation built out of the Jacobi parameters. Then

Theorem 6.1. [7] *Let $z_n^{\pm}(x)$ be the zeros in $[x,\infty)$ and $(-\infty,x)$ closest to x. Then*

$$z_n^{+}(x) - z_n^{-}(x) \geq \left(\sum_{j=0}^{n-1} \|T_j(x)\|^2 \right)^{-1} \tag{6.1}$$

Remarks. 1. If $\sup_n \|T_n(x)\| < \infty$, we get an $O(1/n)$ bound.

2. [7] has an interesting application of this bound which shows that in ergodic cases, Poisson statistics for zeros implies zero Lyapunov exponents.

7. Zeros of Random POPUC

In this section, we will discuss Stoiciu's results on zeros of random POPUC. Fix $r \in (0,1)$ once and for all. Our random space, Ω, will have a sequence

$\alpha_0(\omega), \dots, \alpha_n(\omega), \dots$ of numbers in $\mathbb{D}$, independent identically distributed random variables (iidrv) with uniform density in $\{z \mid |z| \le r\}$ and a sequence in $\partial\mathbb{D}$, $\beta_0(\omega), \beta_1(\omega), \dots$ of iidrv with density uniform on $\partial\mathbb{D}$. The α's and β's are independent of each other.

$\{z_j^{(n)}(\omega)\}_{j=1}^n$ is some listing of the zeros of the OPUC with Verblunsky coefficients $\{\alpha_j(\omega)\}_{j=0}^\infty$ and $\{\tilde{z}_j^{(n)}(\omega)\}_{j=1}^n$ of the zeros of the POPUC with $\{\alpha_j(\omega)\}_{j=0}^{n-2}$ and $\beta_{n-1}(\omega)$. We use

$$\#\{j \mid z_j^{(n)}(\omega) \in S\} \tag{7.1}$$

to denote the number of zeros in a set (counting multiplicity) and $\mathbb{P}(\cdot)$ to denote the probability of some event which we will list as a series of conditions. A final notation:

$$S(\theta_0; a, b) = \{z \in \overline{\mathbb{D}} \mid z \ne 0, \arg z \in (\theta_0 + a, \theta_0 + b)\} \tag{7.2}$$

We can now state Stoiciu's main result [9, 10]:

Theorem 7.1. *For any $r \in (0,1)$, any $\theta_0 \in [0, 2\pi)$, any $a_1 < b_1 \le a_2 < \cdots < b_\ell$, and any $k_1, \dots, k_\ell \in \{0, 1, \dots\}$, we have that*

$$\begin{aligned} \text{Prob}\biggl(\#\biggl(j \biggm| \tilde{z}_j^{(n)}(\omega) \in S\biggl(\theta_0; \frac{2\pi a_p}{n}, \frac{2\pi b_p}{n}\biggr)\biggr) = k_p \text{ for } p = 1, \dots, \ell\biggr) \\ \to \prod_{p=1}^{\ell} \frac{(b_p - a_p)^{k_p}}{k_p!}\, e^{-(b_p - a_p)} \end{aligned} \tag{7.3}$$

This says the zeros are asymptotically Poisson distributed with the same asymptotic distribution as n uniformly randomly distributed points! It is the opposite of clock spacing.

The proof uses the strategy of Minami [17] (and earlier, Molchanov [16]) with rather different tactics. One part is an *a priori* bound of the form

$$\text{Prob}(\#(j \mid z_j^{(n)}(\omega) \in S(\theta_0; a, b)) \ge 2) \le C[n(b-a)]^2 \tag{7.4}$$

Minami gets this from a mysterious determinant cancellation, relying on rank one perturbations. Because the analogous POPUC perturbations are rank two, it is not clear how to extend his proof, so Stoiciu instead uses a Prüfer angle argument.

It is important to understand the other part of his argument since overcoming its limitations is key to handling the OPUC case. Using ideas of Aizenman–Molchanov [46], Aizenman [47], and del Rio *et al.* [48] and their partial extension to OPUC by Simon [49], Stoiciu proves that the eigenfunctions of the unitary CMV matrices associated to POPUC are exponentially localized near centers.

He then considers the unitary matrix obtained from an $N \times N$ CMV matrix by replacing the $\log N$ Verblunsky coefficients at spacing $N/\log N$ by β's. Such a matrix breaks into $\log N$ blocks of size $N/\log N$. A trial function argument and the exponential localization result shows that for all but $O((\log N)^2)$ zeros of the Nth order POPUC, there are eigenvalues of the decoupled matrix very close to the zeros.

Since $(N^{-1}N/\log N)^2 \cdot \log N \to 0$, for N large, by (7.4), the eigenvalues in $S(\theta_0; \frac{2\pi a_p}{n}, \frac{2\pi b_p}{n})$ come from different blocks which are independent, so one gets the classical situation where the Poisson distribution arises.

The starting point of the next section will be the key step above, where a trial function argument is used.

That completes what I have to say about Stoiciu's work [9, 10]. I want to complete this section with a brief mention of some work in preparation by Killip–Stoiciu [50]. As background, we recall some results of [2]. Let $\gamma \in (0,1)$, $C > 0$, and

$$\tilde{\alpha}_j(\omega) = C^\gamma (n+C)^{-\gamma} \alpha_j(\omega) \tag{7.5}$$

be random decaying Verblunsky coefficients. Then the spectral properties have a transition at $\gamma = \frac{1}{2}$ with dependence on C at $\gamma = \frac{1}{2}$:

(a) (Theorem 12.7.1 of [2]) If $\gamma > \frac{1}{2}$, the corresponding measures are purely a.c. for a.e. ω.
(b) (Theorem 12.7.5 of [2]) If $\gamma < \frac{1}{2}$, the corresponding measures are pure point for a.e. ω.
(c) (Theorem 12.7.7 of [2]) If $\gamma = \frac{1}{2}$, then there is pure point spectrum if $\frac{1}{2}Cr^2 > 1$, and otherwise purely singular spectrum with Hausdorff dimension $1 - \frac{1}{2}Cr^2$.

Killip–Stoiciu [50] have tentatively shown results on zeros with a similar structure: If $\gamma > \frac{1}{2}$, there is clock behavior; if $\gamma < \frac{1}{2}$, there is Poisson behavior; and at $\gamma = \frac{1}{2}$, depending on C and r, there are β-distributions intermediate between clock and Poisson.

8. Zeros of Random OPUC

As we noted, the control of random POPUC depends on trial functions, that is, a linear variational principle of the form

$$\operatorname{dist}(z, \operatorname{spec}(A)) \leq \|(A-z)\varphi\| \tag{8.1}$$

if $\|\varphi\| = 1$. This bound holds for normal operators, as can be seen by the spectral theorem. Zeros of POPUC are eigenvalues of a unitary matrix, so (8.1) applies. Since $\pi_n M_z \pi_n$ is not normal, (8.1) does not apply to OPUC.

Linear variational principles — even with a constant on the right side — do not hold for general non-normal matrices and z.

Example 8.1. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\varphi_\varepsilon = (1, \varepsilon)^t/(1+\varepsilon^2)^{1/2}$. Then $\text{dist}(\varepsilon, \text{spec}(A)) = \varepsilon$ but $\|(A-\varepsilon)\varphi\| \leq \varepsilon^2$. In general, for $n \times n$ matrices, one can only hope for bounds if $\|(A-z)\varphi\|$ is replaced by $\|(A-z)\varphi\|^{1/n}$ which gains no smallness from exponential decay.

It was this lack that led Stoiciu to focus on POPUC. Davies–Simon [8] realized that by adding an additional condition valid in the case of OPUC, one can get a linear variational principle:

Theorem 8.1. [8] *If* $|z| \geq \|A\|$, *A is an $n \times n$ matrix, and* $\|\varphi\| = 1$, *then*

$$\text{dist}(z, \text{spec}(A)) \leq \frac{4n}{\pi} \|(A-z)\varphi\| \tag{8.2}$$

Remarks. 1. [8] proves (8.2) where $4n/\pi$ is replaced by $\cot(\pi/4n)$, which is shown to be the optimal constant. Using $\cot(x) \leq 1/x$, one gets (8.2).

2. The proof of (8.2) is not hard. By shifting to a Schur basis, one can suppose A is upper triangular, and by scaling, that $z = 1$ and $\|A\| \leq 1$. Since $\inf\{\|B\varphi\| \mid \|\varphi\| = 1\} = \|B^{-1}\|^{-1}$, (8.2) is equivalent to

$$\|(1-A)^{-1}\| \leq \frac{4n}{\pi} \text{dist}(1, \text{spec}(A))^{-1} \tag{8.3}$$

Letting $C = (1-A)^{-1} + (1-A^*)^{-1} - 1$, and noting $C \geq 0$, one has $|C_{ij}| \leq |C_{ii}|^{1/2}|C_{jj}|^{1/2}$. But for $i < j$, $C_{ij} = [(1-A)^{-1}]_{ij}$ since A is upper triangular. It follows that

$$\text{dist}(1, \text{spec}(A))|[(1-A)^{-1}]_{ij}| \leq \begin{cases} 2 & i < j \\ 1 & i = j \\ 0 & i > j \end{cases}$$

This easily implies (8.3) with $2n$ rather than $4n/\pi$, and with some more work, $\cot(\pi/4n)$.

Once one has (8.2), one uses the localized states for test functions to get zeros of OPUC very close to POPUC and uses (7.4) to prove that with probability 1, for large n, the POPUC zeros are far enough apart that these zeros are distinct. The net result are the following three theorems of Davies–Simon [8]:

Theorem 8.2. *For any $r \in (0,1)$ with probability* 1,

$$\limsup_{n\to\infty} (\log n)^{-2} \#(j \mid |z_j^{(n)}(\omega)| < 1 - n^{-k}) < \infty$$

for any k.

Theorem 8.3. *For any $r \in (0,1)$, any θ_0, a, b, and any $\varepsilon > 0$, we have that, with probability* 1, *for large n, all $z_j^{(n)}(\omega)$ in $S(\theta_0; \frac{2\pi a}{n}, \frac{2\pi b}{n})$ have $|z_j^{(n)}(\omega)| > 1 - \exp(-n^{(1-\varepsilon)})$.*

Theorem 8.4. *For any $r \in (0,1)$, any $\theta_0 \in [0, 2\pi)$, any $a_1 < b_1 \leq a_2 < \cdots < b_\ell$, and any $k_1, \dots, k_\ell \in \{0, 1, \dots\}$, we have that*

$$\mathrm{Prob}\biggl(\#\biggl(j \biggm| \tilde{z}_j^{(n)}(\omega) \in S\biggl(\theta_0; \frac{2\pi a_p}{n}, \frac{2\pi b_p}{n}\biggr)\biggr) = k_p \text{ for } p = 1, \dots, \ell\biggr) \to \prod_{p=1}^{\ell} \frac{(b_p - a_p)^{k_p}}{k_p!} e^{-(b_p - a_p)}$$

Note that Theorems 8.4 and 7.1 are almost the same, but in Theorem 7.1, $|\tilde{z}_j^{(n)}| = 1$, while in Theorem 8.4, $1 - \exp(-n^{(1-\varepsilon)}) \leq |z_j^{(n)}| < 1$.

The zeros of orthogonal polynomials continue to provide beautiful mathematics.

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